STABILITY OF AN EXPANDING VISCOELASTIC SHELL SUBJECT TO AGING

N. Kh. Arutyunyan, M. N. Mikhailov, and V. D. Potapov UDC 539.376

In [1, 2] there is a discussion of the stability over finite and infinite time intervals for compressed expanding inhomogeneously aging viscoelastic rods. Stability is understood in the sense of stability of motion for a dynamic system after Chetaev and Lyapunov. It has been shown that the rod growth mechanism and the growth characteristics have substantial effects on the strain-state parameters and on the critical time.

Here we examine the stability of a compressed expanding thin-walled shell in which the viscoelastic material shows aging. The equations of state are described by ones from the theory of viscoelasticity for inhomogeneously aging bodies [3]. The shell is subject to its own weight and to external loads varying in time.

1. Expanding-Shell Model. Consider a thin-walled shell whose geometrical parameters vary in time because the material grows. We distinguish a surface S in the shell, which may be the median surface or equidistant from it. The positions of points in the shell are defined by the coordinates α_i, z (i = 1, 2), with the curvilinear coordinates α_i reckoned along the lines of principal curvature of surface S, while z is reckoned along the normal to that surface directed toward the centers of positive curvature in the coordinate surface (from the concave side of S if it is elliptical).

We assume that material points with coordinates α_i are first generated on S, after which the walls of the shell are formed. At each instant, the shell is bounded by the edge Γ, which consists of individual parts Γ_i, where α_i = const (i = 1, 2). If S is the internal (external) surface of the shell, growth on the wall occurs along the exterior (interior) normal. If, on the other hand, S occupies an intermediate position, the wall grows along the interior and exterior normals.

This mechanism does not exclude the case where part of the wall thickness is generated at the same time as the corresponding part of S, while the subsequent growth is along the exterior or interior normal, or the two simultaneously. The last case occurs, for example, in strengthening (reconstructing) shells.

2. Equations of Motion for a Growing Viscoelastic Shell. Consider a point with coordinates α_i and z, which is generated at time $t = \tau^*(\rho)$, where $\rho = \{\alpha_i, z\}$. At $\tau^*(\rho)$, the displacements at points on the coordinate surface with coordinates α_i are $u_i^* = u_i(\tau^*(\rho), \rho^0)$, $w^* = w(\tau^*(\rho), \rho^0)$, where $\rho^0 = \{\alpha_1, \alpha_2\}$, with u_i the displacement along coordinate axis i and w the additional shell deflection (displacement along the normal to the coordinate surface). If $t \geq \tau^*(\rho)$, the displacements of this point on the coordinate surface are $u_i = u_i(t, \rho^0)$, $w = w(t, \rho^0)$. If a material particle with coordinates ρ is unstressed at the time of generation $\tau^*(\rho)$, the strains there for $t \geq \tau^*(\rho)$ can be found by means of the modified Kirchhoff–Love hypothesis:

$$\varepsilon_{ij} = \Delta e_{ij} - z\Delta\chi_{ij},$$
$$\Delta e_{ij} = e_{ij}(t, \rho^0) - e_{ij}(\tau^*(\rho), \rho^0);\ \Delta\chi_{ij} = \chi_{ij}(t, \rho^0) - \chi_{ij}(\tau^*(\rho), \rho^0). \quad (2.1)$$

where $\Delta e_{ij} = e_{ij}(t, \rho^0) - e_{ij}(\tau^*(\rho), \rho^0)$; $\Delta\chi_{ij} = \chi_{ij}(t, \rho^0) - \chi_{ij}(\tau^*(\rho), \rho^0)$.

The strains e_{ij}, the curvatures, and the torsion χ_{ij} of the coordinate surface S are defined by expressions from shell theory [4]. The increments Δe_{ij}, $\Delta\chi_{ij}$ should satisfy the equations of continuity for the deformation of surface S, which in the case of small deflections take the form

$$(A_2\Delta\chi_{22})_{,1} - (A_1\Delta\chi_{12})_{,2} - A_{2,1}\Delta\chi_{11} - A_{1,2}\Delta\chi_{12} + \frac{1}{R_1}\Big[2(A_1\Delta e_{12})_{,2} -$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 151-160, March-April, 1986. Original article submitted January 17, 1985.

 0021-8944/86/2702-0300$12.50

$$- (A_2\Delta e_{22})_{,1} + A_{2,1}\Delta e_{11} + \frac{2R_1}{R_2} A_{1,2}\Delta e_{12}\Big] = 0,$$

$$(A_1\Delta\chi_{11})_{,2} - (A_2\Delta\chi_{12})_{,1} - A_{1,2}\Delta\chi_{22} - A_{2,1}\Delta\chi_{12} +$$

$$+ \frac{1}{R_2}\left[2(A_2\Delta e_{12})_{,1} - (A_1\Delta e_{11})_{,2} + A_{1,2}\Delta e_{22} + \frac{2R_2}{R_1} A_{2,1}\Delta e_{12}\right] = 0,$$

$$\frac{\Delta\chi_{11}}{R_2} + \frac{\Delta\chi_{22}}{R_1} + \frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_1}\left\{\frac{1}{A_1}[(A_2\Delta e_{22})_{,1} - (A_1\Delta e_{12})_{,2} - A_{2,1}\Delta e_{11} - A_{1,2}\Delta e_{12}]\right\} +$$

$$+ \frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_2}\left\{\frac{1}{A_2}[(A_1\Delta e_{11})_{,2} - (A_2\Delta e_{12})_{,1} - A_{1,2}\Delta e_{22} - A_{2,1}\Delta e_{12}]\right\} = 0,$$

where A_i and R_i are the Lamé coefficients and radii of principal curvature of S; the subscript after the comma denotes differentiation with respect to the corresponding coordinate α_i.

In particular, the equations of joint strain amount to a single equation for a smooth shell ($A_1 = A_2 = 1$)

$$\Delta e_{11,22} + \Delta e_{22,11} - 2\Delta e_{12,12} = -\frac{1}{R_1}[w(t, \rho^0) - w(\tau^*(\rho), \rho^0)]_{,22} - \frac{1}{R_2}[w(t, \rho^0) - w(\tau^*(\rho), \rho^0)],$$

where

$$\Delta e_{11} = [u_1(t, \rho^0) - u_1(\tau^*(\rho), \rho^0)]_{,1} - \frac{1}{R_1}[w(t, \rho^0) - w(\tau^*(\rho), \rho^0)],$$

$$\Delta e_{22} = [u_2(t, \rho^0) - u_2(\tau^*(\rho), \rho^0)]_{,2} - \frac{1}{R_2}[w(t, \rho^0) - w(\tau^*(\rho), \rho^0)],$$

$$2\Delta e_{12} = [u_1(t, \rho^0) - u_1(\tau^*(\rho), \rho^0)]_{,2} + [u_2(t, \rho^0) - u_2(\tau^*(\rho), \rho^0)]_{,1}.$$

The equations of equilibrium for the expanding shell are as for a nonexpanding one [4], but it is necessary to bear in mind that

$$N_{11}(t, \rho^0) = \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \sigma_{11}\left(1 - \frac{z}{R_2}\right) dz, \ldots,$$

$$M_{11}(t, \rho^0) = \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \sigma_{11}\left(1 - \frac{z}{R_2}\right) z\,dz, \ldots,$$

where $z_+(t, \rho^0)$, $z_-(t, \rho^0)$ are the coordinates of points belonging to the inner and outer surfaces of the shell as measured along the normal to S.

For a thin-walled shell, one usually assumes that

$$N_{ij}(t, \rho^0) \approx \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \sigma_{ij}\,dz, \quad M_{ij}(t, \rho^0) \approx \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \sigma_{ij} z\,dz.$$

The relations between the stresses and strains are taken in the form

$$\sigma_{11} = A_{1111}\varepsilon_{11} + A_{1122}\varepsilon_{22}, \ \sigma_{22} = A_{2211}\varepsilon_{11} + A_{2222}\varepsilon_{22}, \ \sigma_{12} = A_{1212}\varepsilon_{12},$$

where [3]

$$A_{ijkl}\varepsilon_{kl} = E_{ijkl}(t - \tau^*(\rho))\,\varepsilon_{kl}(t, \rho) - \int_{\tau^*(\rho)}^{t} R_{ijkl}(t - \tau^*(\rho), \tau - \tau^*(\rho))\,\varepsilon_{kl}(\tau, \rho)\,d\tau.$$

To derive the forces N_{ij} and M_{ij}, it is necessary to calculate integrals for the form

$$J = \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \left[E_{ijkl}(t - \tau^*(\rho))\,\varepsilon_{kl}(t, \rho) - \int_{\tau^*(\rho)}^{t} R_{ijkl}(t - \tau^*(\rho), \tau - \tau^*(\rho))\,\varepsilon_{kl}(\tau, \rho)\,d\tau\right] dz.$$

We substitute from (2.1) and perform some transformations to get

$$J = E_{ijkl}F(t,\rho^0)e_{kl}(t,\rho^0) - \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} E_{ijkl}(t-\tau^*(\rho))e_{kl}(\tau^*(\rho),\rho^0)\,dz - \tag{2.2}$$
$$- E_{ijkl}S(t,\rho^0)\chi_{kl}(t,\rho^0) + \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} E_{ijkl}(t-\tau^*(\rho))\chi_{kl}(\tau^*(\rho),\rho^0)z\,dz -$$
$$- \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \int_{\tau^*(\rho)}^{t} R_{ijkl}(t-\tau^*(\rho),\tau-\tau^*(\rho))\{e_{kl}(\tau,\rho^0) - e_{kl}(\tau^*(\rho),\rho^0) -$$
$$- [\chi_{kl}(\tau,\rho^0) - \chi_{kl}(\tau^*(\rho),\rho^0)]z\}\,dz\,d\tau,$$

where

$$E_{ijkl}F(t,\rho^0) = \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} E_{ijkl}(t-\tau^*(\rho))\,dz;\quad E_{ijkl}S(t,\rho^0) = \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} E_{ijkl}(t-\tau^*(\rho))z\,dz.$$

As

$$R(t,\tau) = \frac{\partial L(t,\tau)}{\partial\tau},\quad L(t,\tau) = E(\tau) - T(t,\tau)$$

[$E(\tau)$ is the modulus for elastic instantaneous deformation, $T(t, \tau)$ is the relaxation measure $T(t, t) = T(\tau, \tau) = 0$], we write

$$\int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \int_{\tau^*(\rho^0)}^{t} R_{ijkl}(t-\tau^*(\rho),\tau-\tau^*(\rho))e_{kl}(\tau^*(\rho),\rho^0)\,d\tau\,dz =$$
$$= \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} e_{kl}(\tau^*(\rho),\rho^0)[E_{ijkl}(t-\tau^*(\rho)) - L_{ijkl}(t-\tau^*(\rho),0)]\,dz.$$

One can similarly represent the same integral containing $\chi_{k\ell}(\tau^*(\rho), \rho^0)z$.

Then (2.2) becomes

$$J = E_{ijkl}F(t,\rho^0)e_{kl}(t,\rho^0) - E_{ijkl}S(t,\rho^0)\chi_{kl}(t,\rho^0) - \tag{2.3}$$
$$- \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \int_{\tau^*(\rho)}^{t} R_{ijkl}(t-\tau^*(\rho),\tau-\tau^*(\rho))[e_{kl}(\tau,\rho^0) - z\chi_{kl}(\tau,\rho^0)]\,d\tau\,dz -$$
$$- \int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} L(t-\tau^*(\rho),0)[e_{kl}(\tau^*(\rho),\rho^0) - z\chi_{kl}(\tau^*(\rho),\rho^0)]\,dz.$$

It is readily shown [2] that

$$\int_{z_-(t,\rho^0)}^{z_+(t,\rho^0)} \int_{\tau^*(\rho)}^{t} R_{ijkl}(t-\tau^*(\rho),\tau-\tau^*(\rho))[e_{kl}(\tau,\rho^0) - z\chi_{kl}(\tau,\rho^0)]\,d\tau\,dz = \int_{\tau_1^*(\rho)}^{t} [\widetilde{R}^{(0)}_{ijkl}(t,\tau)e_{kl}(\tau,\rho^0) - \widetilde{R}^{(1)}_{ijkl}(t,\tau)\chi_{kl}(\tau,\rho^0)]\,d\tau,$$

where

$$\widetilde{R}^{(\gamma)}_{ijkl}(t,\tau) = \int_{\tau_1^*(\rho)}^{\tau} R_{ijkl}(t-\tau^*(\rho),\tau-\tau^*(\rho))z^{\gamma}(\tau^*(\rho))\,dz(\tau^*(\rho)),\quad \gamma = 0.1;$$

and $\tau_1^*(\rho^0)$ is the instant of generation for the material particle on S with coordinates α_i.

We finally get (2.3) as

$$J = \tilde{L}^{(0)}_{ijkl}(t, t, \rho^0)\, e_{kl}(t, \rho^0) - \tilde{L}^{(1)}_{ijkl}(t, t, \rho^0)\, \chi_{kl}(t, \rho^0) - \tag{2.4}$$

$$- \int\limits_{\tau_1^*(\rho^0)}^{t} \left[\frac{\partial}{\partial\tau}\tilde{L}^{(0)}_{ijkl}(t, \tau, \rho^0)\, e_{kl}(\tau, \rho^0) - \frac{\partial}{\partial\tau}\tilde{L}^{(1)}_{ijkl}(t, \tau, \rho^0)\, \chi_{kl}(\tau, \rho^0)\right] d\tau,$$

where

$$L^{(\gamma)}_{ijkl}(t, \tau, \rho^0) = \int\limits_{\tau_1^*(\rho^0)}^{\tau} L_{ijkl}(t-\xi, \tau-\xi)\, z^{\gamma}(\xi)\, dz(\xi), \quad \gamma = 0,1.$$

With the symbols

$$L^{(\gamma)}_{ijkl}(t, t, \rho^0)\, f_{kl}(t, \rho^0) - \int\limits_{\tau_1^*(\rho^0)}^{t} \frac{\partial}{\partial\tau}\tilde{L}^{(\gamma)}_{ijkl}(t, \tau, \rho^0)\, f_{kl}(\tau, \rho^0)\, d\tau \equiv \tilde{L}^{(\gamma)}_{ijkl} f_{kl} \quad (\gamma = 0, 1, 2),$$

we rewrite (2.4) as

$$J = \tilde{L}^{(0)}_{ijkl} e_{kl} - \tilde{L}^{(1)}_{ijkl} \chi_{kl}.$$

The forces and moments correspondingly are

$$N_{11}(t, \rho^0) = \tilde{L}^{(0)}_{1111} e_{11} + \tilde{L}^{(0)}_{1122} e_{22} - \tilde{L}^{(1)}_{1111} \chi_{11} - \tilde{L}^{(1)}_{1122} \chi_{22}, \ \ldots,$$
$$N_{12}(t, \rho^0) = \tilde{L}^{(0)}_{1212} e_{12} - \tilde{L}^{(1)}_{1212} \chi_{12},$$
$$M_{11}(t, \rho^0) = \tilde{L}^{(1)}_{1111} e_{11} + \tilde{L}^{(1)}_{1122} e_{22} - \tilde{L}^{(2)}_{1111} \chi_{11} - \tilde{L}^{(2)}_{1122} \chi_{22}, \ \ldots,$$
$$M_{12}(t, \rho^0) = \tilde{L}^{(1)}_{1212} e_{12} - \tilde{L}^{(2)}_{1212} \chi_{12}.$$

The boundary conditions at the edges of the shell are written on the basis of the possible changes in the boundaries of the region occupied by S. The solution to the equation system satisfying the corresponding boundary conditions at the edges describes the unperturbed motion.

3. Equations of Perturbed Motion for an Expanding Viscoelastic Shell. We assume that the initial curvature of the coordinate surface, the external loads, and so on have small perturbations δw^0, δq_i, δq_z; the motion due to these is called perturbed. The displacements and the internal forces in that motion differ from those in the unperturbed case by the perturbations δu_i, δw, δN_{ij}, δM_{ij}, δQ_i.

The equations of equilibrium and joint deformation in the coordinate surface under perturbed motion have the same form as for the unperturbed case if one replaces N_{ij}, M_{ij}, Q_i, Δe_{ij}, $\Delta\chi_{ij}$ by $N_{ij} + \delta N_{ij}$, $M_{ij} + \delta M_{ij}$, $Q_i + \delta Q_i$, $\Delta e_{ij} + \delta\Delta e_{ij}$, $\Delta\chi_{ij} + \delta\Delta\chi_{ij}$; from these we subtract the equations corresponding to the unperturbed motion to get equations for the unperturbed motion in a growing shell in relation to permanently acting perturbations.

We further assume that the dimensions of the shell tend to a limit as time passes; as regards the elastic moduli $E_{ijk\ell}(t)$ and relaxation kernels $R_{ijk\ell}(t, \tau)$, we assume that they satisfy

$$\lim_{t\to\infty} E_{ijkl}(t) = E^0_{ijkl} = \text{const}, \ \lim_{\substack{t>\tau \\ \tau\to\infty}} R_{ijkl}(t, \tau) = R_{ijkl}(t-\tau),$$

$$0 \leqslant \int\limits_0^{\infty} R_{ijkl}(\tau)\, d\tau < E^0_{ijkl}.$$

If the external loads are sufficiently small and also tend to limiting values over time, it can be shown that the displacements in the unperturbed motion $u_i(t, \rho^0)$, $w(t, \rho^0)$ and the forces $M_{ij}(t, \rho^0)$, $N_{ij}(t, \rho^0)$ tend to the constant quantities $u_i(\rho^0)$, $w(\rho^0)$, $M_{ij}(\rho^0)$, $N_{ij}(\rho^0)$.

To examine the stability over an infinite interval, we use linearized equations for the perturbed motion. On solving the problem in displacements, these equations may be written in operator form as

$$A\delta u = \delta f, \quad \delta u = \{\delta u_i, \delta w\},$$

where A is the operator in the linearized treatment and δf is a vector dependent on the initial perturbations δw^0, δq_i, ...

The shell will be stable over an infinite interval (Lyapunov stable) if the external-load parameter does not exceed the value corresponding to the degeneracy condition for operator A^0, which is the limit for A. Operator A^0 is formulated, as in the elastic case, by replacing the displacements in the unperturbed motion by their limiting values $u_i(\rho^0)$, $w(\rho^0)$, while the instantaneous elastic moduli $E_{ijk\ell}{}^0$ are replaced by the long-time ones

$$E_{ijkl}^{\text{lt}} = E_{ijkl}^0 - \int_0^\infty R_{ijkl}(\tau)\, d\tau.$$

It is particularly important to examine the stability of the unperturbed motion over a finite interval in order to evaluate the behavior of an expanding viscoelastic shell because of the elevated sensitivity to any imperfections and especially to the initial curvatures of the median surface. Here there are various possible formulations. We consider two of them.

1. We are given a finite time interval [0, T]. We have to find the critical values of the parameters determining the growth (for example, the growth rates, the law followed by the load over time, etc.) for which the maximal values of the perturbations (such as δw) do not exceed the preset value Δ:

$$\sup_{\rho^0} |\delta w(t, \rho^0)| < \Delta, \; t \in [0, T].$$

2. The limiting permissible values Δ for the displacement perturbations are known. We have to find the time t_*, called the critical time, when the maximal displacement perturbation first becomes Δ.

4. Variational Formulation. In some cases, it is best to conduct the study from variational principles. We introduce the Lagrange functional [5]

$$E = \int_{V(t)} \left[\frac{1}{2} E_{ijkl}(t - \tau^*(\rho))\, \varepsilon_{ij}(t, \rho)\, \varepsilon_{kl}(t, \rho) - \varepsilon_{ij}(t, \rho) \int_{\tau^*(\rho)}^{t} R_{ijkl}(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varepsilon_{kl}(\tau, \rho)\, d\tau \right] dV - \Pi,$$

where Π is the potential of the external loads:

$$\Pi = \int_{S(t)} (q_i u_i + q_3 w) A_1 A_2\, d\alpha_1 d\alpha_2 + \int_{\Gamma_1(t)} \left(N_{1i}^* u_i + Q_1^* w + M_{11}^* \omega_1 \right) A_2\, d\alpha_2 + \int_{\Gamma_2(t)} \left(N_{2i}^* u_i + Q_2^* w + M_{22} \omega_2 \right) A_1\, d\alpha_1;$$

with $N_{ij}{}^*$, $Q_i{}^*$, $M_{ii}{}^*$ the external forces applied to the contour Γ_i, while ω_i are the angles of rotation of the normal to S on Γ_i.

The condition for the functional to be stationary is that the first derivative with respect to the displacements is zero at the current instant:

$$\delta E = 0. \tag{4.1}$$

We represent the displacements as expansions in terms of a complete set of functions satisfying the geometrical boundary conditions, and from (4.1) we get a system of decision equations in the expansion coefficients, which are functions of time.

When one examines the stability for the unperturbed motion of a shell subject to conservative loads, one formulates the functional E^*, which is derived from E by replacing the displacements u_i and w corresponding to the unperturbed motion by $u_i + \delta u_i$, $w + \delta w$. The external perturbations are incorporated by introducing instead of the functions w^0, q_i, q_3, Q_i^*, ... the expressions $w^0 + \delta w^0$, $q_i + \delta q_i$, $q_3 + \delta q_3$, $Q_i^* + \delta Q_i^*$, ..., so E^* can be represented as

$$E^* = E + \delta E + \delta^2 E + \dots,$$

where δE and $\delta^2 E$ are terms containing respectively the first degrees and the products of the perturbations δu_i, δw, ...

As we have (4.1), we write

$$E^* = E + \delta^2 E + \dots$$

The stationarity condition for E^* is as before, that the first variation is zero:

$$\delta E^* = 0,$$

where the term corresponding to the unperturbed motion is not varied. Then

$$\delta(\delta^2 E + \dots) = 0. \tag{4.2}$$

From (4.2) we can derive equations and boundary conditions for the boundary-value problem corresponding to the perturbed motion in terms of perturbations.

5. Expanding Cylindrical Shell. Consider a circular cylindrical shell whose expansion occurs along the generator, while the material can also expand symmetrically with respect to the median surface. The shell is subject to its own weight and to an axial compressive load P(t), which is uniformly distributed along the free edge. The shell has an external permanently acting perturbation in the form of an initial axisymmetric curvature of the median surface, which is specified as

$$w^0(x_1) = \delta w^0(x_1) = -c + \left(c + \lambda c x_1 + \sum_{i=2}^{n} a_i x_1^i\right) \exp(-\lambda x_1),$$

where c, a_i, and λ are constants.

The coordinate x_1 is reckoned along the generator for the median surface, while the origin lies at the lower end of the shell.

The strains in axisymmetric deformation for small deflections are given by

$$\varepsilon_{11}(t, \rho) = u_{1,1}(t, x_1) - u_{1,1}(\tau^*(\rho), x_1) - [w_{,11}(t, x_1) - w_{,11}(\tau^*(\rho), x_1)]z,$$
$$\varepsilon_{22}(t, \rho) = -\frac{1}{R}[w(t, x_1) - w(\tau^*(\rho), x_1)],$$

where $\tau^*(\rho)$ is the instant of generation for a material particle having coordinates x_1 and z.

The shell material is isotropic, viscoelastic, and inhomogeneously aging, with the Poisson ratio μ constant over time. Then

$$\sigma_{11} = \frac{1}{1-\mu^2} E(\varepsilon_{11} + \mu\varepsilon_{22}), \quad \sigma_{22} = \frac{1}{1-\mu^2} E(\varepsilon_{22} + \mu\varepsilon_{11}),$$

where

$$E\psi = E(t - \tau^*(\rho))\,\psi(t, x_1) - \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\,\psi(\tau, x_1)\,d\tau.$$

Example 1. Consider a shell of constant length ℓ. At the initial instant ($t_0 = t$) the thickness is constant at h_0. During the subsequent growth, the expansion is symmetrical with respect to the median plane, with the thickness remaining the same at all points, being h(t). The material is viscoelastic and does not show aging, where the expressions are

$$E(t) = E = \text{const}, \; R(t, \tau) = \gamma c \exp(-\gamma(t - \tau)).$$

The shell is compressed by an axial load P(t). The edges have hinged support, which provides for a moment-free state of strain on compression of an ideal shell.

We assume that the median surface has a small axisymmetric curvature $w^0(x_1) = f^0[\sin(m\pi/\ell)]x_1$; the equation in terms of the additional deflection $w(t, x_1)$ is written as

$$D(t)\frac{\partial^4 w(t, x_1)}{\partial x_1^4} - \int_0^t \frac{\partial}{\partial \tau} L^{(2)}(t, \tau)\frac{\partial^4 w(\tau, x_1)}{\partial x_1^4} d\tau + \frac{Eh(t)}{R^2} w(t, x_1) - \\ - \frac{1}{R^2}\int_0^t \frac{\partial}{\partial \tau} L^{(0)}(t, \tau) w(\tau, x_1) d\tau + P(t)\frac{\partial^2}{\partial x_1^2}\left[w(t, x_1) + w^0(x_1)\right] = 0, \tag{5.1}$$

where

$$L^{(2)}(t, \tau) = \frac{h^3(\tau)}{12(1-\mu^2)}\omega(t, \tau); \; L^{(0)}(t, \tau) = h(\tau)\,\omega(t, \tau);$$

$$\omega(t, \tau) = E - C[1 - \exp(-\gamma(t - \tau))]; \; D(t) = \frac{Eh^3(t)}{12(1-\mu^2)}.$$

We seek the deflection $w(t, x_1)$ as

$$w(t, x_1) = f(t)[\sin(m\pi/l)]x_1. \tag{5.2}$$

Then from (5.1) we have

$$[1 - \alpha(t)]f(t) - \frac{1}{N(t)}\int_0^t \left[\frac{m^2\pi^2}{l^2}\frac{\partial L^{(2)}(t, \tau)}{\partial \tau} + \frac{l^2}{m^2\pi^2}\frac{\partial L^{(0)}(t, \tau)}{\partial \tau}\right] f(\tau)d\tau = \alpha(t)f^0. \tag{5.3}$$

Here

$$N(t) = \frac{m^2\pi^2}{l^2}D(t) + \frac{Eh(t)}{m^2\pi^2R^2}; \; \alpha(t) = \frac{P(t)}{N(t)}.$$

The motion corresponding to the moment-free state is called unperturbed. Equation (5.3) describes the change in the deflection amplitude in the perturbed motion.

We assume that $\lim_{t\to\infty} h(t) = h_\infty$, $\lim_{t\to\infty} P(t) = P_\infty$, and from (5.3) it is evident that the unperturbed motion is stable at infinity in relation to perturbation f^0 if the following conditions are obeyed [6]:

$$\alpha(t) < 1 \; \forall t \geqslant 0, \; \alpha_{lt} < 1,$$

where

$$\alpha_{lt} = P_\infty/N_{lt}, \; N_{lt} = \frac{m^2\pi^2}{l^2}D_{lt} + \frac{E_{lt}h_\infty l^2}{m^2\pi^2R^2},$$

$$E_{lt} = E - C, \; D_{lt} = \frac{E_{lt}h_\infty^3}{12(1-\mu^2)}.$$

We further consider an elastic shell (C = 0). Equation (5.1) in that case is written as

$$D(t)\frac{\partial^4 w(t, x_1)}{\partial x_1^4} - \frac{E}{4(1-\mu^2)}\int_0^t h^2(\tau)\dot{h}(\tau)\frac{\partial^4 w(\tau, x_1)}{\partial x_1^4} d\tau + \frac{Eh(t)}{R^2} w(t, x_1) - \\ - \frac{E}{R^2}\int_0^t \dot{h}(\tau) w(\tau, x_1) d\tau + P(t)\frac{\partial^2}{\partial x_1^2}\left[w(t, x_1) + w^0(x_1)\right] = 0 \tag{5.4}$$

(an overdot denotes a derivative with respect to time τ). Then (5.4) differs essentially from the analogous equation for a nonexpanding elastic shell.

From (5.2) and (5.4) we get

$$f(t)+f^0=\frac{f^0}{1-\alpha(0)}\exp\left[\int_0^t \frac{\alpha(\tau)}{1-\alpha(\tau)}\frac{\dot P(\tau)}{P(\tau)}\,d\tau\right]. \tag{5.5}$$

We assume that the compressive force varies in time as follows:

$$P(t)=\varkappa h^n(t).$$

Then for example for a plate ($R\to\infty$) at time t corresponding to the thickness attaining $h(t)=\sqrt{2}h_0$, we have from (5.5) that ($\xi=(f(t)+f^0)/f^0$)

$$\text{for } n=1\ \xi=\sqrt{\frac{1-\alpha(t)}{(1-\alpha(0))^3}},\ \alpha(t)=\frac{\alpha(0)}{2},$$

$$\text{for } n=3\ \xi=\frac{1}{1-\alpha(0)}2^{\frac{3\alpha(t)}{2(1-\alpha(t))}},\ \alpha(t)=\alpha(0).$$

Figure 1 shows the variation of ξ with $\alpha(0)$ for an expanding plate (solid lines) and a nonexpanding one (dashed curves). In the latter case, ξ is defined by

$$\xi=1/(1-\alpha(t)).$$

Figure 2 shows the dependence of ξ on $\eta=h^2(t)/h^2(0)$ for $n=1$ for a growing plate (curve 1) in the case $\alpha(0)=0.5$. For comparison, we show the dependence of ξ on η for a nonexpanding plate with the same initial $\alpha(0)$ (curve 2).

Figures 1 and 2 indicate that the deflection in certain cases of a growing plate may be considerably larger than that in a nongrowing one even for the elastic case and elastic constants unvarying over time. Analogous results apply for shells. The effects of the growth are even more pronounced when the material is viscoelastic.

Example 2. Consider a circular cylindrical shell whose lower end is rigidly clamped, while the other edge is free. To analyze the axisymmetric deformed state, we use Lagrange's variational principle. Then (4.1) becomes

$$\frac{1}{1-\mu^2}\int_0^{l(t)}\left\{\left[\tilde L^{(2)}(t,t,x_1)\,\dot w_{,11}(t,x_1)-\int_{\tau_1^*(x_1)}^{t}\frac{\partial}{\partial\tau}\tilde L^{(2)}(t,\tau,x_1)\,\dot w_{,11}(\tau,x_1)\,d\tau\right]\delta w_{,11}(t,x_1)+\right.$$

$$+\frac{1}{R^2}\left[\tilde L^{(0)}(t,t,x_1)\,w(t,x_1)-\int_{\tau_1^*(x_1)}^{t}\frac{\partial}{\partial\tau}\tilde L^{(0)}(t,\tau,x_1)\,w(\tau,x_1)\,d\tau\right]\delta w(t,x_1)-$$

$$\left.-\frac{\mu}{R}\left[\tilde L^{(0)}(t,t,x_1)\,u_{1,1}(t,x_1)-\int_{\tau_1^*(x_1)}^{t}\frac{\partial}{\partial\tau}\tilde L^{(0)}(t,\tau,x_1)\,u_{1,1}(\tau,x_1)\,d\tau\right]\delta w(t,x_1)\right\}dx_1-$$

$$-\int_0^{l(t)}\left(P(t)+\int_{x_1}^{l(t)}ph(t,x_1)\,dx_1\right)\left[w_{,1}(t,x_1)+w^0_{,1}(x_1)\right]\delta w_{,1}(t,x_1)\,dx_1=0,$$

$$\frac{1}{1-\mu^2}\int_0^{l(t)}\left\{\left[\tilde L^{(0)}(t,t,x_1)u_{1,1}(t,x_1)-\int_{\tau_1^*(x_1)}^{t}\frac{\partial}{\partial\tau}\tilde L^{(0)}(t,\tau,x_1)\,u_{1,1}(\tau,x_1)\,d\tau\right]\delta u_{1,1}(t,x_1)-\right.$$

$$\left.-\frac{\mu}{R}\left[\tilde L^{(0)}(t,t,x_1)\,w(t,x_1)-\int_{\tau_1^*(x_1)}^{t}\frac{\partial}{\partial\tau}\tilde L^{(0)}(t,\tau,x_1)\,w(\tau,x_1)\,d\tau\right]\delta u_{1,1}(t,x_1)\right\}dx_1+ \tag{5.6}$$

$$+\int_0^{l(t)} ph(t, x_1)\,\delta u_1(t, x_1)\,dx_1 + P(t)\,\delta u_1(t, x_1)\big|_{x_1=l(t)} = 0. \tag{5.6}$$

Here $\ell(t)$ is the length of the shell at time t and p is the density of the material.

The deflection w and the axial displacement u_1 are put as sums:

$$w(t, x_1) = \sum_{i=2}^{m} a_i(t)\,x_1^i,\quad u_1(t, x_1) = \sum_{j=1}^{k} b_j(t)\,x_1^j.$$

We determine the coefficients $a_i(t)$, $b_j(t)$ by means of (5.6), in which the integrals with respect to the length are calculated by means of Simpson's quadrature formula, while the integral Volterra equations of the second kind in $a_i(t)$, $b_j(t)$ are solved numerically by the Krylov–Bogolyubov method [7].

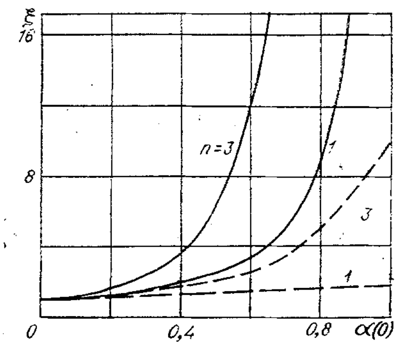

Fig. 1

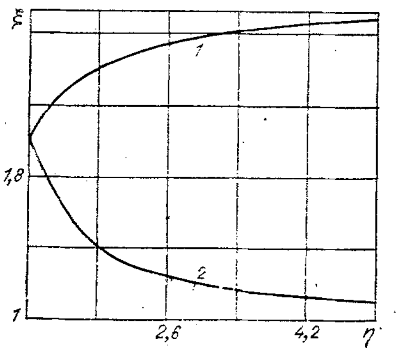

Fig. 2

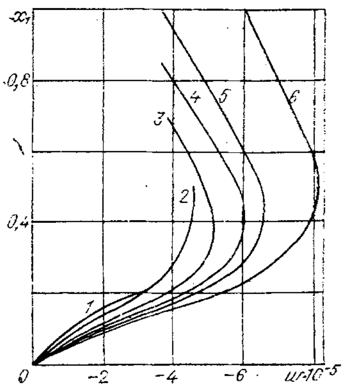

Fig. 3

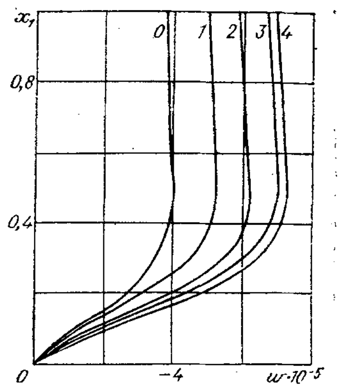

Fig. 4

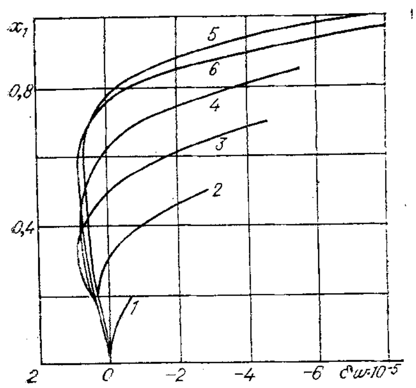

Fig. 5

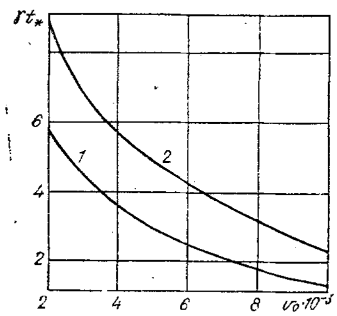

Fig. 6

We further consider a shell growing only along the generator (the wall thickness h is constant over time and along the length). The material expands at a constant rate v_0 over the interval [0, T], after which the length ℓ_0 remains unchanged. We assume that the shell is subject only to a uniformly distributed axial load $P(t) = P_0 = \text{const}$ applied to the free edge.

We assume that

$$R(t, \tau) = -\frac{\partial}{\partial\tau}\{\omega(\tau)[1 - e^{-\gamma(t-\tau)}]\}, \; E(t) = E_0 = \text{const},$$

where $\omega(\tau)$ is a function characterizing the aging. We take this function as

$$\omega(\tau) = C_0 + A_0 e^{-\beta\tau},$$

where C_0, A_0, β, and γ are constants.

Figures 3-6 show calculations for a shell with the following characteristics: $R/\ell_0 = 1$, $h/\ell_0 = 0.04$, $C_0/E_0 = 0.075$, $A_0/E_0 = 0.75$, $P_0/E_0h = 2.5 \cdot 10^{-4}$, $\gamma = 0.02$ day^{-1}, and $\beta = 0.025$ day^{-1}.

Figure 3 shows the positions of the generators for the median surface in unperturbed motion ($w^0 \equiv 0$) at the times γt = 0.4, 1, 1.4, 1.7, 2, and 3.3 (curves 1-6).

The growth rate is $v_0 = 0.01$ day^{-1}, T = 100 days. The negative values for the deflection correspond to the median surface being displaced toward the exterior normal. For comparison, Fig. 4 shows the position of the generators for a nongrowing shell (length ℓ_0) loaded by the same P_0 (curves 0-4 corresponding to γt = 0, 0.4, 0.8, 1.2, 1.4). Figures 3 and 4 show that the growth has a substantial effect not only on w but also on the variation with this over the length.

Figure 5 shows the variation in δw along the length for various instants during the growth. The external perturbation was taken as an axisymmetric curvature of the median surface of the form $w^0(x_1) = 0.01x^2$. Curves 1-6 are for γt = 0.4, 1, 1.4, 1.7, 2, and 3.3. The variation in the deflection perturbations in time and along the length are notable.

Figure 6 shows curves illustrating the dependence of the dimensionless critical time γt_* on the growth rate for two limiting values of the deflection perturbation: $\Delta_1 = 4 \cdot 10^{-5}$ (curve 1) and $\Delta_2 = 8 \cdot 10^{-5}$ (curve 2), which show that the growth rate has a considerable effect on γt_* and thus on the stability.

LITERATURE CITED

1. N. Kh. Arutyunyan and V. D. Potapov, "Stability in a growing viscoelastic rod subject to aging," Dokl. Akad. Nauk SSSR, 270, No. 4 (1983).
2. N. Kh. Arutyunyan, M. N. Mikhailov, and V. D. Potapov, "Stability in a growing viscoelastic reinforced rod subject to aging," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1984).
3. N. Kh. Arutyunyan, "Some problems in creep theory for inhomogeneously aging bodies," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1976).
4. A. S. Vol'mir, Stability in Deformable Systems [in Russian], Nauka, Moscow (1967).
5. Yu. N. Rabotnov, Elements of the Inheritance Mechanics of Solids [in Russian], Nauka, Moscow (1977).
6. A. D. Drozdov, V. B. Kolmanovskii, and V. D. Potapov, "Stability of a rod consisting of an inhomogeneously aging viscoelastic material," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 2 (1984).
7. L. V. Kantorovich and V. I. Krylov, Approximate Methods in Higher Analysis [in Russian], Fizmatgiz, Leningrad (1962).